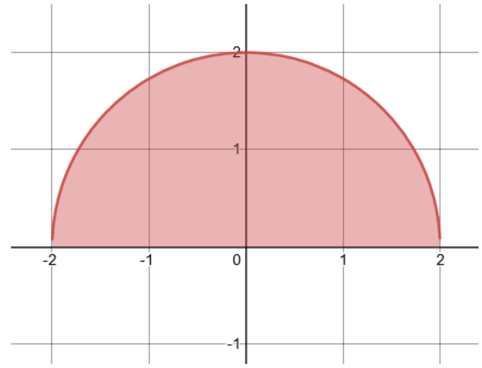
Name: Date:

7B: Trigonometric Substitution

Consider This Integral: $\int \sqrt{4 - x^2}\, dx$

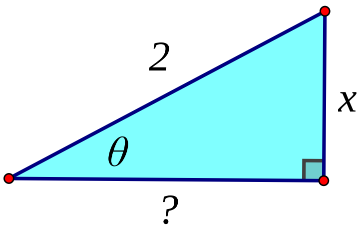
Do you recognize the function $y = \sqrt{4 - x^2}$? This can also be written as $x^2 + y^2 = 4$. This is a circle, or really half of a circle because the function is the positive square root.



Can we use u −substitution or integration by parts? Unfortunately, no!

So, we need something else.

Whenever we see an expression in the form of $\sqrt{x^2 \pm u^2}$, we should consider the Pythagorean Theorem and right triangles.



Try this:

a) What is an expression for length of the side of this triangle?

b) Write an equation relating θ, x, and 2. Then solve it for x and solve it for θ.

c) Now write an equation for the $\cos\theta$, the adjacent length $\sqrt{4 - x^2}$, and the hypotenuse length 2. Then solve this for $\sqrt{4 - x^2}$

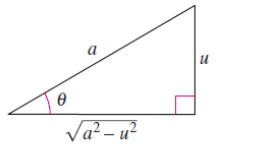
d) Now use trigonometric substitution to evaluate $\int \sqrt{4 - x^2} dx$

TRIGONOMETRIC SUBSTITUTION ($a > 0$)

1. For integrals involving $\sqrt{a^2 - u^2}$, let

$$u = a \sin \theta.$$

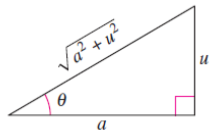
Then $\sqrt{a^2 - u^2} = a \cos \theta$, where $-\pi/2 \le \theta \le \pi/2$.



2. For integrals involving $\sqrt{a^2 + u^2}$, let

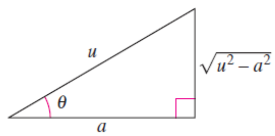
$$u = a \tan \theta.$$

Then $\sqrt{a^2 + u^2} = a \sec \theta$, where $-\pi/2 < \theta < \pi/2$.



3. For integrals involving $\sqrt{u^2 - a^2}$, let

$$u = a \sec \theta.$$



Then

$$\sqrt{u^2 - a^2} = \begin{cases} a \tan \theta \text{ if } u > a, \text{ where } 0 \le \theta < \pi/2 \\ -a \tan \theta \text{ if } u < -a, \text{ where } \pi/2 < \theta \le \pi \end{cases}$$

<u>**Example:**</u> Evaluate

$$\int \frac{dx}{\sqrt{4x^2 + 1}}$$

<u>**Example:**</u> Evaluate

$$\int \frac{\sqrt{x^2 - 3}}{x}\, dx$$